

THE DISTRIBUTION OF THE LOGARITHM OF ORTHOGONAL AND SYMPLECTIC L -FUNCTIONS

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ABSTRACT. We consider the logarithm of the central value $\log L(\frac{1}{2})$ in the orthogonal family $\{L(s, f)\}_{f \in H_k}$ where H_k is the set of weight k Hecke-eigen cusp form for $SL_2(\mathbb{Z})$, and in the symplectic family $\{L(s, \chi_{8d})\}_{d \asymp D}$ where χ_{8d} is the real character associated to fundamental discriminant $8d$. Unconditionally, we prove that the two distributions are asymptotically bounded above by Gaussian distributions, in the first case of mean $-\frac{1}{2} \log \log k$ and variance $\log \log k$, and in the second case of mean $\frac{1}{2} \log \log D$ and variance $\log \log D$. Assuming both the Riemann and Zero Density Hypotheses in these families we obtain the full normal law in both families, confirming a conjecture of Keating and Snaith.

1. INTRODUCTION

An important problem in analytic number theory is to understand the distribution of values of L -functions on the central line $\Re(s) = \frac{1}{2}$. Selberg [12] famously proved that as t varies in large intervals $t \in [T, 2T]$, the real and imaginary parts of the logarithm of Riemann's zeta function become distributed like independent Gaussian random variables. Since that work, there have been several efforts to extend the result to a more general setting. A few years later, Selberg himself [11] proved that for a fixed value of t the imaginary part of $\log L(\frac{1}{2} + it; \chi)$ becomes normally distributed as χ varies among Dirichlet characters to a large prime modulus q . More recently, Bombieri and Hejhal [1] have shown that Selberg's result for zeta is true for the values $\{L(\frac{1}{2} + it)\}_{t \in [T, 2T]}$ of a quite general L -function, under mild assumptions about the zeros of the function, and Wenzhi Luo [9] has verified this condition for the L -function associated to any fixed modular form for $SL_2(\mathbb{Z})$.

Following the ground-breaking work of Katz and Sarnak [5], we now understand the central values $L(\frac{1}{2} + it)$ of an L -function as belonging in a family with a symmetry type governed by one of the classical compact groups. The cases considered thus far, of a fixed L -function with argument high in the critical strip, and of central values of Dirichlet L -functions with varying character of fixed conductor, arise as unitary families; on the basis of calculations from random matrix theory, Keating and Snaith [6] have proposed Selberg-type conjectures for the logarithms of central values of L -functions from families of orthogonal and symplectic symmetry type, as well. These conjectures appear far from reach, however, because they involve only the real part of the logarithm of L -functions at the fixed point $s = \frac{1}{2}$; even the best known analytic methods have thus far only succeeded in proving that a positive proportion of L -functions in a family are non-zero at a single point, and even in the few special cases where the central value is known to be positive, the real part of the logarithm is highly sensitive to the 'low-lying' zeros, near $\frac{1}{2}$, which

cannot presently be controlled. Nonetheless, it is the purpose of this paper to consider what partial results can be established theoretically in two such cases.

Let S_k , $k \equiv 0 \pmod{2}$ be the space of weight k modular cusp forms for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ and let H_k be it's basis of $\sim \frac{k}{12}$ simultaneous eigenvectors of the Hecke operators, normalized to have first Fourier coefficient equal to 1. Let $f \in H_k$ have Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} \lambda_f(n) e(nz);$$

the L -function $L(s, f)$ associated to f is then

$$(1) \quad L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1}.$$

This has completed L -function

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma(s + \frac{k-1}{2}) L(s, f),$$

which satisfies the self-dual functional equation

$$\Lambda(s, f) = i^k \Lambda(1-s, f).$$

When $k \equiv 2 \pmod{4}$ this means that the central value $L(\frac{1}{2}; f) = 0$, so for $k \equiv 0 \pmod{4}$ we consider the family of values $\{L(\frac{1}{2}, f)\}_{f \in H_k}$, which is expected to have orthogonal symmetry type. These central values have a certain extra significance because Kohnen and Zagier [7] proved the striking formula

$$L(\frac{1}{2}, f) = \frac{\pi^k}{(k-1)!} \frac{\langle f, f \rangle}{\langle g, g \rangle}$$

relating the central value $L(\frac{1}{2}, f)$ to the ratio of the Petersson norms of f and a half-integral weight form g that lifts to f under the Shimura correspondance. A particular consequence is that $L(\frac{1}{2}, f) > 0$; this is essentially the only family in which positivity of the central value is known.

As a second example we let $d > 0$ be a fundamental discriminant with associated quadratic character $\chi_d(n) = (\frac{d}{n})$ of conductor d . The corresponding Dirichlet L -function is

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} = \prod_p \left(1 - \frac{\chi_d(p)}{p^s} \right)^{-1}, \quad \Re(s) > 1$$

with completed L -function

$$\Lambda(s, \chi_d) = \left(\frac{8d}{\pi} \right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi_{8d}).$$

This also satisfies the self-dual functional equation

$$\Lambda(s, \chi_d) = \Lambda(1-s, \chi_d)$$

and conjecturally $L(\frac{1}{2}, \chi_{8d}) > 0$, but this is not known. For convenience we consider the family of central values $\{L(\frac{1}{2}, \chi_{8d})\}_{d \in s(D)}$ where $s(D)$ denotes the set of squarefree and odd d , $\frac{D}{2} < d \leq D$; this is expected to be a family exhibiting symplectic symmetry.

We have two primary results. The first result proves, unconditionally, ‘one-half’ of the Keating-Snaith conjectures.

Corollary 1.1. *Let $k \equiv 0 \pmod{4}$. As $k \rightarrow \infty$ we have*

$$\mathbb{P} \left[f \in H_k : \frac{1}{\sqrt{\log \log k}} \left(\log L\left(\frac{1}{2}, f\right) + \frac{1}{2} \log \log k \right) > A \right] \leq \frac{1}{\sqrt{2\pi}} \int_A^\infty e^{-\frac{x^2}{2}} dx + o_A(1).$$

In particular, for any fixed $\epsilon > 0$, $L(\frac{1}{2}, f) < (\log k)^{-1/2+\epsilon}$ with probability $1 - o_\epsilon(1)$. Also, as $D \rightarrow \infty$,

$$\mathbb{P} \left[d \in s(D) : \frac{1}{\sqrt{\log \log D}} \left(\log |L(\frac{1}{2}, \chi_{8d})| - \frac{1}{2} \log \log D \right) > A \right] \leq \frac{1}{\sqrt{2\pi}} \int_A^\infty e^{-\frac{x^2}{2}} dx + o_A(1).$$

In [13], Soundararajan made the basic observation that, on the Riemann Hypothesis, while zeros near $\frac{1}{2} + it$ can greatly alter the value of $\log |\zeta(\frac{1}{2} + it)|$, they always decrease its value as compared with that of $\log |\zeta(\frac{1}{2} + \sigma + it)|$ at points off the critical line. Our proof of Corollary 1.1 is based upon an unconditional version of this fact, together with the following slightly technical result.

Theorem 1.2. *Let $\sigma = \sigma(k)$ be a function of k , tending to 0 as $k \rightarrow \infty$ in such a way that $\sigma \log k \rightarrow \infty$ but $\frac{\sigma \log k}{\sqrt{\log \log k}} \rightarrow 0$. Also, for $f \in H_k$ put*

$$A(f) = \frac{1}{\sqrt{\log \log k}} \left(\log |L(\frac{1}{2} + \sigma, f)| + \frac{1}{2} \log \log k \right).$$

Then

$$\frac{1}{|H_k|} \sum_{f \in H_k} \delta_{A(f)} \rightarrow N(0, 1), \quad k \rightarrow \infty.$$

Here δ_x is the point mass at x , $N(0, 1)$ is the standard normal distribution, and the convergence is in the sense of distributions.

Similarly, let $\sigma = \sigma(D)$ be a function of D , tending to 0 as $D \rightarrow \infty$ in such a way that $\sigma \log D \rightarrow \infty$ but $\frac{\sigma \log D}{\sqrt{\log \log D}} \rightarrow 0$. For $d \in s(D)$, put

$$A(d) = \frac{1}{\sqrt{\log \log D}} \left(\log |L(\frac{1}{2} + \sigma, \chi_{8d})| - \frac{1}{2} \log \log D \right).$$

Then

$$\frac{1}{|s(D)|} \sum_{d \in s(D)} \delta_{A(d)} \rightarrow N(0, 1), \quad D \rightarrow \infty.$$

This Theorem is proven using Selberg’s method in [11]; in particular it makes use of ‘zero-density’ estimates putting almost all of the low-lying zeros of the corresponding L -functions very near the half-line. In the case of $L(s, \chi_{8d})$, such a result is essentially available from the work of Conrey and Soundararajan in [2]. For the case of $L(s, f)$, this is a concurrent result of the author in [3].

For our second main result we assume some weak conjectural information about the low-lying zeros in the families $\{L(s, f)\}_{f \in H_k}$, and $\{L(s, \chi_{8d})\}_{d \in s(D)}$ in order to deduce the full Keating-Snaith conjectures for these families. Given $f \in H_k$ and s near $\frac{1}{2}$, $L(s, f)$

has conductor $\asymp k^2$, and therefore for $1 \ll T = k^{o(1)}$ the number of zeros of $L(s, f)$ up to height T grows as $\frac{T}{\pi} \log k$. Thus, based upon purely density considerations, we might expect that for most $f \in H_k$, $\gamma_{\min}(f) \gg \frac{1}{\log k}$, where

$$\gamma_{\min}(f) = \min_{\rho = \frac{1}{2} + \beta + i\gamma} |\gamma|$$

is the height of the lowest non-trivial zero of $L(s, f)$. Similarly, for $d \in s(D)$ and s near $\frac{1}{2}$, $L(s, \chi_{8d})$ has conductor $\asymp D$, and therefore we might typically expect that $\gamma_{\min}(d) \gg \frac{1}{\log D}$. We formalize this heuristic in the following hypothesis.

Hypothesis 1.3 (Low-lying Zero Hypothesis). *Assume $y = y(k) \rightarrow \infty$ with k . Then*

$$\mathbb{P} \left[f \in H_k : \gamma_{\min}(f) < \frac{\pi}{y \log k} \right] = o(1), \quad k \rightarrow \infty.$$

Similarly, if $y = y(D) \rightarrow \infty$ with D then

$$\mathbb{P} \left[d \in s(D) : \gamma_{\min}(d) < \frac{2\pi}{y \log D} \right] = o(1), \quad D \rightarrow \infty.$$

In fact, stronger and more detailed statements about the low-lying zeros in these two families are expected to be true. Specifically, Iwaniec, Luo and Sarnak [4] have conjectured that in essentially any natural family of L -functions of conductor C , the one-level density of zeros at a scale of $\frac{2\pi}{\log C}$ depends asymptotically only on the symmetry type of the family. For our two families of L -functions, their 'Zero Density Conjecture' takes the following shape.

Conjecture 1.4 (Zero Density Conjecture). *Let $\phi(x)$ be a Schwarz class function on \mathbb{R} with Fourier transform having compact support. Define the densities*

$$W(SO_{\text{even}})(x)dx = \left(1 + \frac{\sin 2\pi x}{2\pi x}\right) dx, \quad W(Sp)(x)dx = \left(1 - \frac{\sin 2\pi x}{2\pi x}\right) dx$$

and write the non-trivial zeros of $L(s)$ as $\rho = \frac{1}{2} + i\gamma$, with γ possibly complex if the Riemann Hypothesis for $L(s)$ is false. Then

$$\lim_{\substack{k \rightarrow \infty \\ k \equiv 0 \pmod{4}}} \frac{1}{|H_k|} \sum_{f \in H_k} \sum_{\substack{\Lambda(\rho, f)=0 \\ \rho = \frac{1}{2} + i\gamma}} \phi\left(\frac{\gamma \log k}{\pi}\right) = \int_{-\infty}^{\infty} \phi(x) W(SO_{\text{even}})(x) dx$$

and

$$\lim_{D \rightarrow \infty} \frac{1}{|s(D)|} \sum_{d \in s(D)} \sum_{\substack{\Lambda(\rho, \chi_{8d})=0 \\ \rho = \frac{1}{2} + i\gamma}} \phi\left(\frac{\gamma \log D}{2\pi}\right) = \int_{-\infty}^{\infty} \phi(x) W(Sp)(x) dx.$$

It is a straightforward exercise to prove that our Low-lying Zero Hypothesis is implied by the Zero Density Conjecture together with the Riemann Hypothesis for the corresponding family of L -functions.

We now state our second main result.

Theorem 1.5. Suppose the Low-lying Zero Hypothesis holds in the family $\{L(s, f)\}_{f \in H_k}$. For $f \in H_k$ put

$$B(f) = \frac{1}{\sqrt{\log \log k}} \left(\log L\left(\frac{1}{2}, f\right) + \frac{1}{2} \log \log k \right).$$

Then, as distributions

$$\frac{1}{|H_k|} \sum_{f \in H_k} \delta_{B(f)} \rightarrow N(0, 1), \quad k \rightarrow \infty.$$

Similarly, assume the Low-lying Zero Hypothesis in the family $\{L(s, \chi_{8d})\}_{d \in s(D)}$ and for $d \in s(D)$ put

$$B(d) = \frac{1}{\sqrt{\log \log D}} \left(\log |L\left(\frac{1}{2}, \chi_{8d}\right)| - \frac{1}{2} \log \log D \right).$$

Then, in the sense of distributions,

$$\frac{1}{|s(D)|} \sum_{d \in s(D)} \delta_{B(d)} \rightarrow N(0, 1), \quad D \rightarrow \infty.$$

In particular, either of these results is true if both the Riemann Hypothesis and the Zero Density Hypothesis is true for the corresponding family of L -functions.

2. BACKGROUND

In this section we collect together standard facts regarding our two families of L -functions, as well as the part of Selberg's work that we need for our arguments.

2.1. L -function coefficients, and orthogonality. For $f \in H_k$, the Fourier coefficients of f satisfy the Hecke relations

$$(2) \quad \lambda_f(m) \lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

A specific consequence of this fact is that for distinct primes p_1, \dots, p_r we have

$$(3) \quad \lambda(p_1)^{e_1} \lambda(p_2)^{e_2} \dots \lambda(p_r)^{e_r} = \sum_{0 \leq j_1 \leq \lfloor \frac{e_1}{2} \rfloor} \dots \sum_{0 \leq j_r \leq \lfloor \frac{e_r}{2} \rfloor} c(\mathbf{e}, \mathbf{j}) \lambda_f(p_1^{e_1-2j_1} \dots p_r^{e_r-2j_r})$$

for some positive coefficients $c(\mathbf{e}, \mathbf{j})$.

Lemma 2.1. We have $c(\mathbf{2}, *) = 1$ where $\mathbf{2}$ is the string consisting entirely of 2's and $*$ is any string containing 0's and 1's. Also, for general \mathbf{e}, \mathbf{j} , $c(\mathbf{e}, \mathbf{j}) \leq 2^{e_1 + \dots + e_r}$.

Recall, also, Deligne's bound $|\lambda_f(n)| \leq d(n)$.

We use the following basic orthogonality relation on H_k .

Lemma 2.2. *Let $0 < \delta < 2$. There exists $\gamma = \gamma(\delta) > 0$ such that if $m < k^{2-\delta}$ then*

$$\frac{1}{|H_k|} \sum_{f \in H_k} \lambda_f(m) = \frac{\delta_{m=\square}}{\sqrt{m}} + O(k^{-\gamma}).$$

Proof. Actually, this is a combination of two different estimates. Using the Petersson Trace Formula, Rudnick and Soundararajan ([10], Lemma 2.1) prove that for $mn < \frac{k^2}{10000}$,

$$\sum_{f \in H_k} \frac{2\pi^2}{k-1} L(1, \text{sym}^2 f)^{-1} \lambda_f(m) \lambda_f(n) = \delta_{m=n} + O(e^{-k}).$$

Here $w_f = \frac{2\pi^2}{k-1} L(1, \text{sym}^2 f)^{-1}$ is the so-called ‘harmonic weight’ of f , and $L(s, \text{sym}^2 f)$ is the symmetric square L -function attached to $L(s, f)$ with coefficients given by

$$L(s, \text{sym}^2 f) = \sum_{n=1}^{\infty} \frac{\rho_f(n)}{n^s} = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}, \quad \Re(s) > 1.$$

A now-standard method of Kowalski-Michel ([8], Proposition 2) allows the removal of the harmonic weight by truncating the Dirichlet series for $L(1, \text{sym}^2 f)$; with $x = k^{1-\delta/2}$ and recalling $|H_k| \sim \frac{k-1}{12}$, their method gives

$$\begin{aligned} \frac{1}{|H_k|} \sum_{f \in H_k} \lambda_f(m) &= \frac{1}{|H_k|} \sum_{f \in H_k} w_f \frac{k-1}{2\pi^2} L(1, \text{sym}^2 f)^{-1} \lambda_f(m) \\ &= \frac{1}{\zeta(2)} \sum_{f \in H_k} w_f \lambda_f(m) \sum_{\ell^2 d < x} \frac{\lambda_f(d^2)}{\ell^2 d} + O(k^{-\gamma}). \end{aligned}$$

Substituting the bound of Rudnick and Soundararajan, one deduces the lemma. \square

For the real characters χ_{8d} , our basic orthogonality relation is the following.

Lemma 2.3. *Let $n < D^{2-\delta}$. Then there is $\gamma = \gamma(\delta) > 0$ such that*

$$\sum_{d \in s(D)} \left(\frac{8d}{n} \right) = \delta_{n=\square} \prod_{\substack{p|n \\ \text{odd}}} \left(\frac{p}{p+1} \right) + O(D^{-\gamma}).$$

Proof. Note that $\mu(2d)^2$ is exactly the indicator function for odd, squarefree d . Rudnick and Soundararajan ([10] Lemma 3.1) prove, for any $z > 3$, that if n is a perfect square then

$$\sum_{d \leq z} \mu(2d)^2 \left(\frac{8d}{n} \right) = \frac{z}{\zeta(2)} \prod_{p|2n} \left(\frac{p}{p+1} \right) + O(z^{\frac{1}{2}+\epsilon} n^{\epsilon})$$

and if n is not a square then

$$\sum_{d \leq z} \mu(2d)^2 \left(\frac{8d}{n} \right) = \frac{z}{\zeta(2)} \prod_{p|2n} \left(\frac{p}{p+1} \right) + O(z^{\frac{1}{2}} n^{\frac{1}{4}} \log(2n)).$$

The result follows on taking successively $z = D/2, D$. \square

2.2. Selberg's work: two expressions for the logarithm. Writing the Euler product of $L(s, f)$ as

$$L(s, f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s} \right)^{-1} \left(1 - \frac{\overline{\alpha_p}}{p^s} \right)^{-1}, \quad \Re(s) > 1$$

we have that for $m = 1, 2, \dots$

$$\lambda_f(p^m) = \alpha_p^m + \alpha_p^{m-2} + \dots + \alpha_p^{-m+2} + \alpha_p^{-m}$$

where for each p , α_p is a complex number of modulus 1 solving $\alpha_p + \overline{\alpha_p} = \lambda_f(p)$. Logarithmically differentiating $L(s, f)$ term-by-term we obtain

$$(4) \quad -\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s} = \sum_{m=1}^{\infty} \sum_p \frac{(\alpha_p^m + \overline{\alpha_p}^m) \log p}{p^{ms}}, \quad \Re(s) > 1.$$

In particular, $\Lambda_f(n)$ is supported on prime powers, and is given explicitly by

$$(5) \quad \Lambda_f(p^m) = (\lambda_f(p^m) - \lambda_f(p^{m-2})) \log p, \quad m \geq 1,$$

with the convention that $\lambda_f(p^{-1}) = 0$.

Similarly we have

$$L(s, \chi_{8d}) = \prod_p \left(1 - \frac{\chi_{8d}(p)}{p^s} \right)^{-1}, \quad \Re(s) > 1$$

and logarithmically differentiating this leads to

$$(6) \quad -\frac{L'}{L}(s, \chi_{8d}) = \sum_n \frac{\Lambda_{8d}(n)}{n^s}, \quad \Re(s) > 1$$

with Λ_{8d} supported on primes powers and

$$(7) \quad \Lambda_{8d}(p^n) = \left(\frac{8d}{p^n} \right) \log p.$$

In a standard way one may write down an expression for $-\frac{L'}{L}(s)$ similar to (4) and (6) when $\frac{1}{2} < \Re(s) \leq 1$, although in this case the zeros of $L(s)$ enter into the formula. The following lemma is the analog of [12], Lemma 10 with $L(s)$ replacing the Riemann zeta function.

Lemma 2.4. *Let $*$ stand in for either $8d$ or f , so that $L(s, *)$ is either $L(s, \chi_{8d})$ for some $d \in s(D)$ or $L(s, f)$ for some $f \in H_k$.*

Let $x > 1$ be a parameter and define

$$\Lambda_{x,*}(n) = \Lambda_*(n) a_x(n); \quad a_x(n) = \begin{cases} 1, & 1 \leq n \leq x \\ \frac{\log^2 \frac{x^3}{n} - 2 \log^2 \frac{x^2}{n}}{2 \log^2 x}, & x \leq n \leq x^2 \\ \frac{\log^2 \frac{x^3}{n}}{2 \log^2 x}, & x^2 \leq n \leq x^3 \end{cases}.$$

For s not coinciding with a trivial or non-trivial zero of $L(s, *)$ we have

$$(8) \quad -\frac{L'}{L}\left(\frac{1}{2} + s, *\right) = \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2}+s}} - \frac{1}{\log^2 x} \sum_{\substack{\rho: \Lambda(\rho,*)=0 \\ \text{non-trivial}}} \frac{x^{\rho-\frac{1}{2}-s}(1-x^{\rho-\frac{1}{2}-s})^2}{(\frac{1}{2} + s - \rho)^3} \\ - \frac{1}{\log^2 x} \sum_{\substack{q: L(-q,*)=0, \Lambda(-q,*) \neq 0 \\ \text{trivial}}} \frac{x^{-q-\frac{1}{2}-s}(1-x^{-q-\frac{1}{2}-s})^2}{(\frac{1}{2} + q + s)^3}.$$

Proof. The sum $\sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2}+s}}$ is the result of expanding $\frac{L'}{L}(z, *)$ in it's Dirichlet series in

$$\frac{1}{2\pi i \log^2 x} \int_{(3)} \frac{x^{z-\frac{1}{2}-s}(1-x^{z-\frac{1}{2}-s})^2}{(z-\frac{1}{2}-s)^3} \frac{L'}{L}(z, *) dz$$

and integrating term-by-term. The remainder of the expression is obtained by shifting the z -contour leftward and evaluating residues. \square

Put

$$(9) \quad \gamma_f(s) = (2\pi)^{-s} \Gamma(s + \frac{k-1}{2}), \quad \gamma_{8d}(s) = \left(\frac{8d}{\pi}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right)$$

so that we may write in a unified way

$$\Lambda(s, *) = \gamma_*(s) L(s, *)$$

for the completed L -function corresponding to either $L(s, f)$ or $L(s, \chi_{8d})$. The completed L -function is entire of order 1 and hence has a Hadamard product running over it's zeros,

$$(10) \quad \Lambda(s, *) = e^{A+Bs} \prod_{\rho: \Lambda(\rho,*)=0} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

Logarithmically differentiating $\Lambda(s, *)$ one proves the following lemma.

Lemma 2.5. *For real $\sigma > 0$ we have*

$$(11) \quad -\frac{L'}{L}\left(\frac{1}{2} + \sigma, *\right) = \frac{\gamma'_*}{\gamma_*}\left(\frac{1}{2} + \sigma\right) - \sum_{\rho}' \frac{1}{\frac{1}{2} + \sigma - \rho}.$$

Furthermore, the sum over zeros is given by

$$(12) \quad \frac{1}{2} \sum_{\rho=\frac{1}{2}+\beta+i\gamma} \left(\frac{1}{\sigma-\beta+i\gamma} + \frac{1}{\sigma-\beta-i\gamma} \right) = \sum_{\rho=\frac{1}{2}+\beta+i\gamma} \frac{\sigma-\beta}{(\sigma-\beta)^2 + \gamma^2}.$$

One of Selberg's major achievements in [12] was that he gave an efficient way to compute $\log \zeta(\frac{1}{2} + it)$ as a short sum over primes; by balancing the expression for $-\frac{\zeta'}{\zeta}(s)$ coming from the Hadamard product as in (11) against the expression from the Euler product (8), he was able to bound the contribution of the zeros in (8). To do so, Selberg introduced a perturbation $\sigma_{x,t}$ depending on the location of the zeros of ζ near height t , and evaluated $\log \zeta(\frac{1}{2} + \sigma_{x,t} + it)$ in place of $\log \zeta(\frac{1}{2} + it)$.

For $\log |L(\frac{1}{2}, *)|$ the analog of $\sigma_{x,t}$ is

$$(13) \quad \sigma_{x,*} = 2 \max_{\rho \in \mathcal{G}_{x,*}} \left(\beta, \frac{2}{\log x} \right); \quad \mathcal{G}_* = \left\{ \rho = \frac{1}{2} + \beta + i\gamma : |\gamma| \leq \frac{x^{3|\beta|}}{\log x}, |\beta| \geq \frac{2}{\log x} \right\}.$$

Selberg's argument for $\log \zeta(\frac{1}{2} + \sigma_{x,t} + it)$ carries over with trivial modifications to bound the zero sum of $L(s, *)$ at $s = \frac{1}{2} + \sigma_{x,*}$ and thus to the evaluation of $\log L(\frac{1}{2} + \sigma_{x,*}, *)$; the result is the following lemma.

Lemma 2.6. *Let $C = k^2$ for $L(s, f)$ or $C = 8d$ for $L(s, \chi_{8d})$ be the conductor of the L -function near $s = \frac{1}{2}$. We have*

$$(14) \quad \sum_{\substack{\rho = \frac{1}{2} + \beta + i\gamma \\ \Lambda(\rho, *) = 0}} \frac{\sigma_{x,*}}{(\sigma_{x,*} - \beta)^2 + \gamma^2} = O \left(\left| \sum_{n < x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \sigma_{x,*}}} \right| \right) + O(\log C)$$

and

$$(15) \quad \log L(\frac{1}{2} + \sigma_{x,*}, *) = \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \sigma_{x,*}} \log n} + O \left(\frac{1}{\log x} \left| \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \sigma_{x,*}}} \right| \right) + O \left(\frac{\log C}{\log x} \right).$$

Proof. See [12] pp 22-26. □

In order to proceed further with Selberg's approach we need an understanding of the perturbation $\sigma_{x,*}$, that is, we need input regarding the distribution of zeros of $L(s, *)$ near the central point $s = \frac{1}{2}$ as either f varies in H_k or d varies in $s(D)$. Our basic analytic ingredient is the following.

Theorem 2.7. *For a sufficiently small $\delta > 0$ there exists $\theta = \theta(\delta)$ such that, uniformly in $\frac{2}{\log k} < \sigma < \frac{1}{2}$ and $\frac{10}{\log k} < T < k^{2\delta}$,*

$$N(\sigma, T, k) \stackrel{\text{def}}{=} \frac{1}{|H_k|} \sum_{f \in H_k} \# \left\{ L(\frac{1}{2} + \beta + i\gamma, f) = 0 : \sigma < \beta, |\gamma| < T \right\} = O(Tk^{-2\theta\sigma} \log k),$$

and also, uniformly in $\frac{4}{\log D} < \sigma < \frac{1}{2}$ and $\frac{10}{\log D} < T < D^\delta$,

$$N(\sigma, T, D) \stackrel{\text{def}}{=} \frac{1}{|s(D)|} \sum_{d \in s(D)} \# \left\{ L(\frac{1}{2} + \beta + i\gamma, \chi_{8d}) = 0 : \sigma < \beta, |\gamma| < T \right\} = O(TD^{-\theta\sigma} \log D).$$

Proof. The part of the theorem regarding $f \in H_k$ is Theorem 1.1 in [3]. For $d \in s(D)$, this result is proved in [2] for the family $\{L(s, \chi_{-8d})\}_{d \in s(D)}$, in the most difficult range $|T| \ll \frac{1}{\log D}$. The changes needed to adapt this to positive fundamental discriminants are trivial, and it is straightforward to extend their result to $|T| \gg D^\delta$, for instance, along the same lines as in [3]. Alternatively, the reader may take the second statement as a black box. □

As a consequence we derive the following essential lemma.

Lemma 2.8. Let \mathcal{F} be one of the two families of L -functions, either $\mathcal{F} = \{L(s, f)\}_{f \in H_k}$ or $\mathcal{F} = \{L(s, \chi_{8d})\}_{d \in s(D)}$. Denote \mathbb{P} the uniform probability on \mathcal{F} . Let $C = k^2$ or $C = D$ be the respective conductor of the family.

For $x = x(C)$ growing with C in such a way that $\frac{\log x}{\log C} \rightarrow 0$ as $C \rightarrow \infty$ we have

$$\mathbb{P} \left[\exists \rho = \frac{1}{2} + \beta + i\gamma : \Lambda(\rho, *) = 0, \beta > \frac{4}{\log x}, |\gamma| \leq \frac{x^{3\beta}}{\log x} \right] = o(1)$$

as $C \rightarrow \infty$.

Proof. Assume that C is sufficiently large so that $\log x \leq \frac{1}{25} \log C$. We have

$$\begin{aligned} \mathbb{P} \left[\exists \rho : \beta > \frac{4}{\log x}, |\gamma| \leq \frac{x^{3\beta}}{\log x} \right] &\leq \mathbb{P} \left[\bigcup_{j=4}^{\lceil \frac{\log x}{2} \rceil} \left\{ \exists \rho : \beta > \frac{j}{\log x}, |\gamma| \leq \frac{e^{3(j+1)}}{\log x} \right\} \right] \\ &\leq \sum_{j=4}^{\lceil \frac{\log x}{2} \rceil} \mathbb{P} \left[\exists \rho : \beta > \frac{j}{\log x}, |\gamma| \leq \frac{e^{3(j+1)}}{\log x} \right]. \end{aligned}$$

By applying Theorem 2.7, the last sum is bounded by

$$\ll \sum_{j=4}^{\lceil \frac{\log x}{2} \rceil} \frac{e^{3(j+1)}}{\log x} C^{-\frac{\theta_j}{\log x}} \log C \leq \frac{\log C}{\log x} \sum_{j=4}^{\lceil \frac{\log x}{2} \rceil} e^{(-\theta \frac{\log C}{\log x} + 3)j} \ll \frac{\log C}{\log x} e^{\frac{-\theta \log C}{\log x}}$$

and this tends to zero as $C \rightarrow \infty$. □

2.3. Convergence in the sense of distributions. Before turning to the main argument, we record, for repeated later use, the following simple fact concerning convergence in the sense of distributions.

Suppose we have a sequence of finite sets $\{R_n\}$; for each n let there be two functions $f, \tilde{f} : R_n \rightarrow \mathbb{R}$, so that we obtain two sequences of probability measures $\{\mu_n\}, \{\tilde{\mu}_n\}$ on \mathbb{R} ,

$$\mu_n = \frac{1}{|R_n|} \sum_{s \in R_n} \delta_{f(s)}, \quad \tilde{\mu}_n = \frac{1}{|R_n|} \sum_{s \in R_n} \delta_{\tilde{f}(s)}.$$

Lemma 2.9. Let μ be a finite (Borel) measure on \mathbb{R} . Each of the following three conditions is sufficient to guarantee the simultaneous convergence in distribution

$$\mu_n \xrightarrow{d} \mu \quad \Leftrightarrow \quad \tilde{\mu}_n \xrightarrow{d} \mu$$

of μ_n and $\tilde{\mu}_n$ to μ .

$$\begin{aligned}
\text{(i)} \quad & \frac{1}{|R_n|} \sum_{\substack{s \in R_n \\ f(s) \neq \tilde{f}(s)}} 1 = o(1), & n \rightarrow \infty \\
\text{(ii)} \quad & \sup_{s \in R_n} |f(s) - \tilde{f}(s)| = o(1), & n \rightarrow \infty \\
\text{(iii)} \quad & \frac{1}{|R_n|} \sum_{s \in R_n} |f(s) - \tilde{f}(s)|^2 = o(1), & n \rightarrow \infty.
\end{aligned}$$

3. THE DISTRIBUTION OF THE PRIME SUM

We first show that short prime sums ($x = C^{o(1)}$)

$$\left\{ \sum_{n \leq x} \frac{\Lambda_f(n)}{\sqrt{n}} \right\}_{f \in H_k}, \quad \left\{ \sum_{n \leq x} \frac{\Lambda_{8d}(n)}{\sqrt{n}} \right\}_{d \in S(d)}$$

converge to the appropriate Gaussian distributions as the conductor $C \rightarrow \infty$. The main work will then be in comparing $\log |L(\frac{1}{2}, *)|$ to these sums.

Proposition 3.1. *Let $C = k^2$ for the family $\mathcal{F} = \{L(s, f)\}_{f \in H_k}$ and $C = D$ for the family $\{L(s, \chi_{8d})\}_{d \in S(D)}$. Assume $x = x(C)$ grows with C in such a way that $\frac{\log x}{\log C} \rightarrow 0$ as $C \rightarrow \infty$, but $\log \log x = \log \log C + o(\sqrt{\log \log C})$. Define, for $f \in H_k$,*

$$P(f) = \frac{1}{\sqrt{\log \log k}} \left(\sum_{n \leq x} \frac{\Lambda_f(n)}{n^{\frac{1}{2}} \log n} + \frac{1}{2} \log \log k \right),$$

and for $d \in S(D)$,

$$P(d) = \frac{1}{\sqrt{\log \log D}} \left(\sum_{n \leq x} \frac{\Lambda_{8d}(n)}{n^{\frac{1}{2}}} - \frac{1}{2} \log \log D \right).$$

We have

$$(16) \quad \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \delta_{P(*)} \xrightarrow{d} N(0, 1), \quad C \rightarrow \infty.$$

Also, for each C let $\{b_n(C)\}_{n=1}^\infty$ be a sequence of real numbers, bounded independently of C . Then

$$(17) \quad \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \left| \sum_{n < x^3} \frac{\Lambda_*(n) b_n}{n^{\frac{1}{2}}} \right|^2 = O(\log^2 x), \quad C \rightarrow \infty.$$

Proof. We show the proof for the family $\mathcal{F} = \{L(s, f)\}_{f \in H_k}$; the argument for real characters is essentially the same, with the caveat that the positive mean of $L(\frac{1}{2}, \chi_{8d})$ comes from the fact that $\left(\frac{8d}{p^2}\right) = 1$ if $p \nmid 8d$.

For (16), let $f \in H_k$ and write

$$P(f) = \frac{1}{\sqrt{\log \log k}} \left[\sum_{m=1}^{\infty} \frac{1}{m} \sum_{p < x^{\frac{1}{m}}} \frac{\Lambda_f(p^m)}{p^{\frac{m}{2}} \log p} - \frac{1}{2} \log \log k \right].$$

In view of the evaluation of the coefficients $\Lambda_f(n)$ given in (5) we have

$$\begin{aligned} P(f) &= \frac{1}{\sqrt{\log \log k}} \sum_{p < x} \frac{\lambda_f(p)}{p^{\frac{1}{2}}} + \frac{1}{2\sqrt{\log \log k}} \sum_{p < \sqrt{x}} \frac{\lambda_f(p^2) - 1}{p} + \frac{1}{2} \sqrt{\log \log k} + o(1) \\ &= \frac{1}{\sqrt{\log \log k}} \sum_{p < x} \frac{\lambda_f(p)}{p^{\frac{1}{2}}} + \frac{1}{2\sqrt{\log \log k}} \sum_{p < \sqrt{x}} \frac{\lambda_f(p^2)}{p} + o(1), \end{aligned}$$

by Merten's theorem for $\sum \frac{1}{p}$. Now we may assume that k is sufficiently large so that $x^2 < k^{2-\delta}$. Then

$$\begin{aligned} \frac{1}{|H_k|} \sum_{f \in H_k} \left[\sum_{p < \sqrt{x}} \frac{\lambda_f(p^2)}{p} \right]^2 &= \sum_{p_1, p_2 < \sqrt{x}} \frac{1}{p_1 p_2} \frac{1}{|H_k|} \sum_{f \in H_k} \lambda_f(p_1) \lambda_f(p_2) \\ &= \sum_{p < \sqrt{x}} \frac{1}{p^2} + O(k^{-\gamma} \log^2 x) = O(1) \end{aligned}$$

by applying Lemma 2.2. Thus by Lemma 2.9 it suffices to prove

$$(18) \quad \frac{1}{|H_k|} \sum_{f \in H_k} \delta_{\tilde{P}(f)} \xrightarrow{d} N(0, 1); \quad \tilde{P}(f) = \frac{1}{\sqrt{\log \log k}} \sum_{p < x} \frac{\lambda_f(p)}{p^{\frac{1}{2}}}.$$

This we do by the method of moments.

Let m be fixed and assume now that k is sufficiently large so that $x^{2m} < k^{2-\delta}$, $x^m < k^{\frac{\gamma}{2}}$. We have

$$\frac{1}{|H_k|} \sum_{f \in H_k} \left| \sum_{p < x} \frac{\lambda_f(p)}{p^{\frac{1}{2}}} \right|^{2m} = \sum_{p_1, \dots, p_{2m} < x} \frac{1}{\sqrt{p_1 \cdots p_{2m}}} \frac{1}{|H_k|} \sum_{f \in H_k} \lambda_f(p_1) \cdots \lambda_f(p_{2m}).$$

When some p_i appears an odd number of times in the list, we see from the expression (3) that $\lambda_f(p_1) \cdots \lambda_f(p_r)$ can be written as a linear combination of $O_m(1)$ terms $\lambda_f(n_i)$, for which none of the n_i are squares. Thus by Lemma 2.2 the contribution of all such terms is $\ll_m k^{-\frac{\gamma}{2}}$.

Among terms containing each p_i an even number of times, those containing some p_i at least 4 times contribute $\ll_m (\log \log x)^{m-2}$, which is an error term. We are left to consider terms containing each prime exactly twice. These contribute

$$O_m(k^{-\gamma} (\log \log k)^m) + \sum_{\substack{p_1, \dots, p_m < x \\ \text{distinct}}} \frac{1}{p_1 \cdots p_m} \sum_{d|p_1 \cdots p_m} \frac{1}{d} = \frac{(2m)!}{2^m m!} (\log \log x)^m (1 + o_m(1)).$$

The claimed convergence in (18) thus follows from the fact that the Gaussian distribution is determined by its moments.

To prove (17), assume $x^6 < \min(k^{2-\delta}, k^\gamma)$ and split the primes, prime squares, and higher powers as

$$\left| \sum_{n < x^3} \frac{\Lambda_f(n) b_n}{n^{\frac{1}{2}}} \right|^2 \leq 3 \left[\left| \sum_{p < x^3} \frac{\lambda_f(p) b_p \log p}{p^{\frac{1}{2}}} \right|^2 + \left| \sum_{p < x^{\frac{3}{2}}} \frac{O(\log p)}{p} \right|^2 + O(1) \right]$$

Thus

$$\begin{aligned} \frac{1}{|H_k|} \sum_{f \in H_k} \left| \sum_{n < x^3} \frac{\Lambda_f(n) b_n}{n^{\frac{1}{2}}} \right|^2 &\leq \frac{3}{|H_k|} \sum_{f \in H_k} \left| \sum_{p < x^3} \frac{\lambda_f(p) b_p \log p}{p^{\frac{1}{2}}} \right|^2 + O(\log^2 x) \\ &\leq 3 \sum_{p_1, p_2 \leq x^3} \frac{b_{p_1} b_{p_2} \log p_1 \log p_2}{\sqrt{p_1 p_2}} \frac{1}{|H_k|} \sum_{f \in H_k} \lambda_f(p_1) \lambda_f(p_2) + O(\log^2 x) \\ &\leq \sum_{p \leq x^3} \frac{O(\log^2 p)}{p} + O(k^{-\gamma/2}) + O(\log^2 x) \\ &= O(\log^2 x). \end{aligned}$$

□

4. PROOF OF MAIN RESULTS

Throughout this section we let \mathcal{F} be a family of L -functions, either $\mathcal{F} = \{L(s, f)\}_{f \in H_k}$ or $\mathcal{F} = \{L(s, \chi_{8d})\}_{d \in s(D)}$, and we let $C = k^2$ or $C = D$ for the conductor in the family. We also let $*$ stand in for the typical element in the family.

Proof of Theorem 1.2. Choose $x = x(C)$ by $\frac{4}{\log x} = \sigma$ and observe that $\frac{\log C}{\log x} \rightarrow \infty$ while $\frac{\log C}{\log x \sqrt{\log \log C}} \rightarrow 0$ as $C \rightarrow \infty$. In particular, $\log \log x = \log \log C + O(\log_3 C)$, fulfilling the conditions of Proposition 3.1, so that $\frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \delta_{P(*)} \xrightarrow{d} N(0, 1)$ as $C \rightarrow \infty$.

Recall that we set

$$A(f) = \frac{1}{\sqrt{\log \log k}} \left(\log |L(\frac{1}{2} + \frac{4}{\log x}, f)| + \frac{1}{2} \log \log k \right)$$

and

$$A(d) = \frac{1}{\sqrt{\log \log D}} \left(\log |L(\frac{1}{2} + \frac{4}{\log x}, \chi_{8d})| - \frac{1}{2} \log \log D \right).$$

By Lemma 2.8 there is a set $E \subset \mathcal{F}$ of measure $o(1)$ such that outside E , $\sigma_{x,*} = \frac{4}{\log x}$. Thus by (15) we have

$$\log L(\frac{1}{2} + \frac{4}{\log x}, *) = \sum_{n < x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}} \log n} + O \left(\frac{1}{\log x} \left| \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}}} \right| \right) + O \left(\frac{\log C}{\log x} \right)$$

except for on a set of measure $o(1)$. Note that the second error term contributes $o(1)$ to $A(*)$. Now

$$\sum_{n < x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}} \log n} = \sum_{n < x} \frac{\Lambda_*(n)}{n^{\frac{1}{2}} \log n} + \sum_{n < x} \frac{\Lambda_*(n)}{n^{\frac{1}{2}} \log n} \left(n^{\frac{-4}{\log x}} - 1 \right) + \sum_{x \leq n < x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}} \log n}.$$

Applying (17) of Proposition 3.1 successively with corresponding choices of b_n , we find that

$$\begin{aligned} \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \left[\frac{1}{\log x} \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}}} \right]^2 &= O(1), & b_n &= a_x(n) \\ \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \left[\sum_{n < x} \frac{\Lambda_*(n)}{n^{\frac{1}{2}} \log n} \left(n^{\frac{-4}{\log x}} - 1 \right) \right]^2 &= O(1), & b_n &= \begin{cases} \frac{\log x}{\log n} \left(n^{\frac{-4}{\log x}} - 1 \right), & n \leq x \\ 0, & x \leq n < x^3 \end{cases} \\ \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \left[\sum_{x \leq n < x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}} \log n} \right]^2 &= O(1), & b_n &= \begin{cases} 0, & n < x \\ a_x(n) \frac{\log x}{\log n}, & x \leq n < x^3 \end{cases}. \end{aligned}$$

Thus by Lemma 2.9,

$$\frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \delta_{P(*)} \xrightarrow{d} N(0, 1) \quad \Rightarrow \quad \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}} \delta_{A(*)} \xrightarrow{d} N(0, 1).$$

□

We will deduce Corollary 1.1 from Theorem 1.2 by applying the following Proposition, which is an analog of the upper bound in the Proposition of [13], in the case when RH for the L -function is not assumed.

Proposition 4.1. *Continue to let $\mathcal{F} = \{L(s, f)\}_{f \in H_k}$ or $\mathcal{F} = \{L(s, \chi_{8d})\}_{d \in s(D)}$, and let C be the conductor of the L -functions in the family. For $\sigma_{x,*}$ as defined in (13) we have*

$$\log |L(\frac{1}{2}, *)| \leq \log L(\frac{1}{2} + \sigma_{x,*}, *) + \sigma_{x,*} \log C + O(1).$$

Proof.

$$\begin{aligned} (19) \quad \log |L(\frac{1}{2}, *)| - \log L(\frac{1}{2} + \sigma_{x,*}, *) &= \int_0^{\sigma_{x,*}} -\frac{L'}{L}(\frac{1}{2} + \sigma, *) d\sigma \\ &= \log \frac{\gamma_*(\frac{1}{2} + \sigma_{x,*})}{\gamma_*(\frac{1}{2})} - \sum_{\rho = \frac{1}{2} + \beta + i\gamma: \Lambda(\rho, *) = 0} \int_0^{\sigma_{x,*}} \frac{\sigma - \beta}{(\sigma - \beta)^2 + \gamma^2} d\sigma \\ &\leq \sigma_{x,*} \log C + O(1) - \frac{1}{2} \sum_{\rho = \frac{1}{2} + \beta + i\gamma: \Lambda(\rho, *) = 0} \log \left(\frac{(\sigma_{x,*} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right) \end{aligned}$$

from the formulas (11) and (12).¹ For those ρ for which $|\beta| \leq \frac{\sigma_{x,*}}{2}$ the logarithm is plainly positive. Among the remaining ρ , pair $\rho = \frac{1}{2} + \beta + i\gamma$ and $\rho' = \frac{1}{2} - \beta + i\gamma$, with, say, $\beta > 0$. Note that by (13), $\rho, \rho' \notin \mathcal{G}_{x,f}$ so that $|\gamma| > \frac{x^{3\beta}}{\log x}$.

The combined contribution of ρ and ρ' to the sum in (19) is

$$\begin{aligned}
 & \log \left[\frac{(\sigma_{x,*} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \cdot \frac{(\sigma_{x,*} + \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right] \\
 &= \log \left[\left(1 + \frac{(\sigma_{x,*} - 2\beta)\sigma_{x,*}}{\beta^2 + \gamma^2} \right) \left(1 + \frac{(\sigma_{x,*} + 2\beta)\sigma_{x,*}}{\beta^2 + \gamma^2} \right) \right] \\
 (20) \quad &= \log \left[1 + \frac{\sigma_{x,*}^2}{\beta^2 + \gamma^2} \left(2 + \frac{\sigma_{x,*}^2 - 4\beta^2}{\beta^2 + \gamma^2} \right) \right]
 \end{aligned}$$

Now observe that

$$\gamma > \frac{x^{3\beta}}{\log x} \geq 3\beta$$

so that

$$(21) \quad 2 \geq 2 + \frac{\sigma_{x,*}^2 - 4\beta^2}{\beta^2 + \gamma^2} \geq 2 - \frac{4\beta^2}{10\beta^2} \geq \frac{8}{5}$$

Hence

$$\text{expr. (20)} \geq \log \left[1 + \frac{8}{5} \frac{\sigma_{x,*}^2}{\beta^2 + \gamma^2} \right] > 0.$$

It follows that the zeros contribute a negative amount to (19), which proves the Proposition. \square

Deduction of Corollary 1.1. Take $x = x(C)$ growing such that $\frac{\log C}{\log x} \rightarrow \infty$ but $\frac{\log C}{\log x \sqrt{\log \log C}} \rightarrow 0$ as $C \rightarrow \infty$, as in the proof of Theorem 1.2. Then as before we have $\sigma_{x,*} = \frac{4}{\log x}$ except for on a set of measure $o(1)$. It follows from Proposition 4.1 that

$$\frac{\log |L(\frac{1}{2}, *)|}{\sqrt{\log \log C}} \leq \frac{\log L(\frac{1}{2} + \frac{4}{\log x}, *)}{\sqrt{\log \log C}} + o(1)$$

except on a set of measure $o(1)$, and the Corollary now follows from the convergence in distribution of the right hand side proved in Theorem 1.2. \square

We now prove Theorem 1.5 by bounding the negative contribution of the zeros in Proposition 4.1 by invoking the Low-lying Zero Hypothesis.

Proof of Theorem 1.5. Let $x = x(C)$ and $y = y(C)$ be parameters growing with C , satisfying the conditions

$$(1) \quad \frac{\log C}{\log x} \rightarrow \infty$$

¹Note that by choice of $\sigma_{x,*}$, $L(s, *)$ has no zero on the real axis for $s = \frac{1}{2} + \sigma$ and $\sigma > \sigma_{x,*}$. In the case that $L(s, f)$ has a zero between $\frac{1}{2}$ and $\frac{1}{2} + \sigma_{x,*}$ the logarithm of $L(s, *)$ at $\frac{1}{2}$ is defined by continuous variation along a path that makes a small semicircle in the upper half plane around the zero, and the line integrals above also follow this path.

$$(2) \frac{\sqrt{\log \log C}}{\log y} \rightarrow \infty, \text{ but } \frac{\log C}{y \log x} \rightarrow 0$$

$$(3) \frac{\log C \log y}{\log x \sqrt{\log \log C}} \rightarrow 0$$

as $C \rightarrow \infty$. For instance, these are simultaneously satisfied with $\log x = \log C (\log \log C)^{-\frac{1}{4}}$, $y = \log \log C$.

Since $\frac{\log C}{\log x} \rightarrow \infty$, $\sigma_{x,*} = \frac{4}{\log x}$ except on a set of measure $o(1)$ in \mathcal{F} . Thus, invoking the Low-lying Zero Hypothesis, there is a set of \mathcal{F}^* of measure $1 - o(1)$ in \mathcal{F} on which $\sigma_{x,*} = \frac{4}{\log x}$ and for all $L(s, *) \in \mathcal{F}^*$, all zeros $\rho = \frac{1}{2} + \beta + i\gamma$ of $\Lambda(s, *)$ satisfy

$$|\gamma| > \frac{1}{y \log x}.$$

Restricting to \mathcal{F}^* , by (19) we have

$$\log |L(\frac{1}{2} + \frac{4}{\log x}, *)| - \log |L(\frac{1}{2}, *)| = O\left(\frac{\log C}{\log x}\right) + \frac{1}{2} \sum_{\rho=\frac{1}{2}+\beta+i\gamma: \Lambda(\rho, *)=0} \log \left(\frac{(\frac{4}{\log x} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right).$$

In view of $\frac{\log C}{\log x} = o(\sqrt{\log \log C})$ it suffices to prove the bound

$$(\dagger) \quad \frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}^*} \left| \sum_{\rho=\frac{1}{2}+\beta+i\gamma: \Lambda(\rho, *)=0} \log \left(\frac{(\frac{4}{\log x} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right) \right|^2 = o(\log \log C)$$

in order to deduce the theorem from Lemma 2.9 together with the normal approximation of $\log L(\frac{1}{2} + \frac{4}{\log x}; *)$ proved in Theorem 1.2.

In the sum over zeros of (\dagger) , for ρ with $|\beta| < \frac{2}{\log x}$ we bound the summand in absolute value by

$$\begin{aligned} \left| \log \left(\frac{(\sigma_{x,*} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right) \right| &= \left| \log \left(1 - \frac{(\sigma_{x,*} - \beta)^2 - \beta^2}{(\sigma_{x,*} - \beta)^2 + \gamma^2} \right) \right| \\ &\leq \left| \log \left(1 - \frac{\sigma_{x,*}^2}{\sigma_{x,*}^2 + \gamma^2} \right) \right| \\ &= \left| \log \left(1 - \frac{1}{1 + (\frac{\gamma \log x}{4})^2} \right) \right| \ll \frac{\log y}{1 + (\frac{\gamma \log x}{4})^2}, \end{aligned}$$

by using $|\gamma| \log x \gg \frac{1}{y}$. Since $|\beta| \leq \frac{2}{\log x}$, the last quantity is bounded by

$$(22) \quad \ll \frac{\log y}{\log^2 x} \frac{1}{(\frac{4}{\log x} - \beta)^2 + \gamma^2}.$$

For $|\beta| > \frac{2}{\log x} = \frac{\sigma_{x,*}}{2}$ we pair the contributions of ρ and $\rho' = \rho - 2\beta$; by (20) this combined contribution is

$$\log \left[1 + \frac{\sigma_{x,*}^2}{\beta^2 + \gamma^2} \left(2 + \frac{\sigma_{x,*}^2 - 4\beta^2}{\beta^2 + \gamma^2} \right) \right].$$

Since $|\beta| > \frac{\sigma_{x,*}}{2}$ we have $\rho, \rho' \notin \mathcal{G}_{x,f}$, and therefore $|\gamma| \geq \frac{x^{3\beta}}{\log x} \geq 3\beta$. By (21),

$$2 \geq 2 + \frac{\sigma_{x,*}^2 - 4\beta^2}{\beta^2 + \gamma^2} \geq \frac{8}{5},$$

and therefore the combined contribution is bounded in absolute value by

$$(23) \quad \ll \frac{\sigma_{x,*}^2}{\beta^2 + \gamma^2} \ll \frac{1}{\log^2 x} \frac{1}{(\frac{4}{\log x} - \beta)^2 + \gamma^2}.$$

Combining (22) and (23),

$$\begin{aligned} \left| \sum_{\rho=\frac{1}{2}+\beta+i\gamma} \log \left(\frac{(\frac{4}{\log x} - \beta)^2 + \gamma^2}{\beta^2 + \gamma^2} \right) \right| &= O \left(\frac{\log y}{\log^2 x} \sum_{\rho} \frac{1}{(\frac{4}{\log x} - \beta)^2 + \gamma^2} \right) \\ &= O \left(\frac{\log y}{\log x} \left| \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}}} \right| \right) + O \left(\frac{\log y \log C}{\log x} \right) \end{aligned}$$

by (14) of Lemma 2.6.

Since

$$\frac{\log y \log C}{\log x} = o(\sqrt{\log \log C}),$$

the bound (†) now follows from

$$\frac{1}{|\mathcal{F}|} \sum_{* \in \mathcal{F}^*} \left| \frac{\log y}{\log x} \sum_{n \leq x^3} \frac{\Lambda_{x,*}(n)}{n^{\frac{1}{2} + \frac{4}{\log x}}} \right|^2 = O(\log^2 y) = o(\log \log C)$$

by (17) of Proposition 3.1. □

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